A Constructive Method for Calculating the Conformal Charge

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We develop high temperature expansions for the conformal charge, c, of the square lattice Ising and 3-state Potts models using Cardy's expression for c in terms of the second moment of the energy-energy correlation function. Using Padé and integral approximant methods to analyze the series, we estimate its value to be 0.500 ± 0.001 and 0.80 ± 0.01 for the Ising and 3-state Potts models respectively. In addition, we show that the conformal charge can be bounded in terms of another universal amplitude ratio.

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One of the most remarkable developments in statistical mechanics in recent years has been the exploitation of conformal symmetries in 2-dimensional critical theories.[1] Much of the early work was focussed on the conformal invariance of the correlations at the critical point itself. Recent work by Zamolodchikov[2] and Cardy[3] has succeeded in extending many of the relevant ideas to correlations away from the critical point. These results have opened up the possibility of calculating the quantities of interest by perturbative methods. In conformal theories, the central quantity, which characterizes different universality classes, is the conformal charge or the conformal anomaly c. Zamolodchikov has derived a C-theorem which is analogous to Boltzmann's H-theorem. He defines a function C which is non-increasing along the renormalization group trajectories and reduces to the conformal charge at the critical points. Using this C-function, Cardy has shown that the conformal charge is expressible as a hyper-universal amplitude ratio.

We begin by considering the moments of the energy-energy correlation function

$$\mu_{E,m} = \frac{1}{N} \sum_i \sum_j r_{ij}^m (\langle e_i e_j \rangle - \langle e_i \rangle \langle e_j \rangle), \tag{1}$$

where the sum runs over all bonds of the lattice, e_i represents the interaction energy between the spins across the bond i, r_{ij} is the distance between the midpoints of the bonds i and j in units of the lattice spacing and N is the total number of sites in the problem. The specific heat is proportional to $\mu_{E,0}$. The central charge at the critical point can be expressed as[3]

$$c = \lim_{K \to K_c} \frac{12\pi (K_c - K)^2}{(2-\alpha)^2} \mu_{E,2} \tag{2}$$

Here, α is the specific heat critical exponent, K is the reduced coupling constant and K_c is its value at the critical point. For $\alpha > 0$, this definition leads to a rather appealing physical interpretation for the central charge. It is, apart from a constant factor, the singular part of the free energy per correlation volume,

$$c = \lim_{K \to K_c} -12\pi \frac{(1-\alpha)}{(2-\alpha)} f_s \xi_E^2. \tag{3}$$

Here f_s is the singular part of the free energy in units of $k_B T_c$ and ξ_E is the correlation length defined via the second moment of the energy-energy correlation function,

$$\xi_E^2 = \mu_{E,2} / \mu_{E,0}. \tag{4}$$

By virtue of hyperscaling, the expression in Eq. (3) goes to a universal constant as one approaches the critical point. From a calculational point of view Eq. (2) is the most convenient. Cardy also showed[3] that it leads to $c = 1/2$ for the Ising model in agreement with known results.

In this paper we develop high temperature expansions for the second moment of the energy-energy correlation function for the Ising and 3-state Potts models. By use of Eq. (2), this expansion leads directly to an expansion for the conformal charge. We find that the series is a rapidly converging one for the Ising model, but that is not so for the Potts case. However, in both cases we can use series extrapolation methods to estimate their value. Using integral approximants[4] we obtain $c = 0.500 \pm 0.001$ and 0.80 ± 0.01 for the two cases, respectively. This is in excellent agreement with the known exact answers of $1/2$ and $4/5$. To our knowledge this represents the first constructive method for calculating the conformal charge from the Hamiltonian. Previously this quantity has been calculated by studying the size dependence of the free energy at the critical point,[5] or by identifying it by searching through the tables of Friedan *et al.*[6]

The partition function for the q-state Potts model on the square lattice is[7]

$$Z = Tr_{s_i} \prod_{\langle i,j \rangle} e^{K \delta_{s_i s_j}}, \tag{5}$$

where the s_i take q different values, δ is a Kronecker delta function and the product runs over every pair of nearest neighbor bonds on the square lattice. The Ising case corresponds to $q = 2$. The natural high temperature expansion variable is

$$v = (e^K - 1)/(e^K - 1 + q). \quad (6)$$

We now briefly discuss the series generation method. Following standard terminology[8] from cluster expansions $\mu_{E,2}$ can be expressed as a sum over distinct graphs of the lattice, of the weight of the graph times the lattice constant of the graph.

$$\mu_{E,2} = \sum_g W(g) \times L(g). \quad (7)$$

Here all graphs which are not related by a lattice symmetry (which for the square lattice includes translation, rotation by $\pi/2$ and reflections about x and y axes) must be treated separately. The weight of a graph can be defined recursively as

$$W(g) = \mu_{E,2}(g) - \sum_{s \subset g} W(s), \quad (8)$$

where $\mu_{E,2}(g)$ is obtained by restricting the sums in Eq. (1) and the product in (5) to the bonds contained in the graph g. It is straightforward to show that (a) the weight of a graph with N_b bonds is of order $v^{N_b - 2}$, and (b) weights of disconnected and articulated graphs are identically zero. Thus, to develop the expansions correctly to order M, one needs to consider all star graphs with $M + 2$ or fewer bonds and calculate $\mu_{E,2}$ for these graphs. Let the expansion be expressed as

$$\mu_{E,2} = \sum_{m=2} a_m v^m. \quad (9)$$

The coefficients a_m are given in Table I.

TABLE I. Expansion coefficients in powers of v for $\mu_{E,2}$ of the q-state Potts models.

order	$q = 2$	$q = 3$
2	2	16/9
3	0	32/9
4	20	16
5	0	544/9
6	162	1312/9
7	0	5056/9
8	1200	15488/9
9	0	4448
10	8462	147824/9
11	0	384064/9
12	57804	1172696/9
13	0	3651824/9
14	386102	

In $d = 2$, the critical point for the q-state Potts models is given by

$$e^{K_c} - 1 = \sqrt{q}, \quad (10)$$

or,

$$v_c = \sqrt{q}/(q + \sqrt{q}). \quad (11)$$

By use of this result, and by making appropriate changes of variables, Eq. (2) becomes

$$c = \frac{12\pi}{(2-\alpha)^2}[\lim_{v \to v_c} \mu_{E,2}(1 - v/v_c)^2]. \quad (12)$$

Note is made that the other K to v conversion factors cancel to unity. For the Ising case the ferromagnet and the antiferromagnet are related by a symmetry, thus the series involves only even powers of v. It is then more convenient to consider the expression,

$$c = \frac{3\pi}{(2-\alpha)^2}[\lim_{v \to v_c} \mu_{E,2}(1 - v^2/v_c^2)^2]. \quad (13)$$

Setting $\alpha = 0$, as is appropriate for the $2d$ Ising model, this leads to the expansion ($x = v/v_c$)

$$c/A = 1 - 0.284271x^2 - 0.047040x^4 - 0.022726x^6 \\ - 0.009963x^8 - 0.005325x^{10} - 0.003219x^{12} - \ldots,$$

with

$$A = 0.808518\ldots.$$

At $x = 1$ this series is rapidly converging. The succesive partial sums lead to estimates for c of 0.579, 0.541, 0.522, 0.514, 0.509, and 0.507 respectively, which is clearly converging to the correct answer of 1/2. In fact, using integral approximants to estimate the value of the series at $x = 1$ we obtain $c = 0.5$ with an uncertainty of 0.001.

From Eq. (12), setting $\alpha = 1/3$ as is appropriate for the 3-state Potts model, we obtain the expansion

$$c/A = 1 - 1.267949x + 0.741670x^2 - 0.012196x^3 \\ - 0.656986x^4 + 0.799704x^5 - 0.352534x^6 \\ - 0.377246x^7 + 0.899853x^8 - 0.920235x^9 \\ + 0.478853x^{10} + 0.109393x^{11} + \ldots,$$

with

$$A = 3.2324629\ldots.$$

This series is not as rapidly converging as the Ising case. However, if we use integral approximants to estimate its value at $x = 1$, we obtain $c = 0.79 \pm 0.02$. We have analyzed the series by several other methods. We can directly estimate the coefficient of $(1 - v/v_c)^{-2}$ in $\mu_{E,2}$. This leads to an estimate for c of 0.80 ± 0.01. To get a more unbiassed estimate, that does not require building in the value of v_c into the series, we consider the expansion for c^* defined by

$$c^*(K) = \frac{48\pi\mu}{(2-\alpha)^2}\left(\frac{\mu}{\frac{\partial\mu}{\partial K}}\right)^2, \qquad \lim_{K\to K_c} c^*(K) = c. \quad (14)$$

Summing up this series using ordinary Padé approximants, we obtain $c = 0.80 \pm 0.01$. Clearly, different methods of analyzing the series agree quite well.

It is interesting to consider the hyper-universal amplitude ratio for arbitrary dimension d, namely the singular part of the free energy per correlation volume, that is to say, ξ_E^d replaces ξ_E^2 in Eq. (3), and for general lattice structures we include a factor of V/a^d where V is the volume per lattice site and a is the lattice spacing. For $d > 4$ hyperscaling fails, and the conformal charge estimator diverges to infinity. For the one dimensional Ising model, it is zero as $f_s = 0$. It would be interesting to compute this quantity for the 3-dimensional Ising model. This effort is left for the future.

In the meantime we can give an upper bound for the central charge c in terms of currently, more widely available quantities than the energy-energy correlation length. We will now demonstate that the spin-spin correlation length (plus unity) is greater than or equal to the energy-energy correlation length. Thus the replacement of ξ_E in Eq. (2) by ξ in $c^*(K)$ will result in an upper bound. We will present this demonstration in the context of the Simon-Griffiths class of spin variables[9], where the results of Aizenman[10] that we use are known (with some additional discussion in spots) to be valid. Let us take $\alpha_{i,j} = 1$ if i and j are nearest neighbors, and zero otherwise. Then if we write out explicitly,

$$r_{i,j,k,l}^2 = \tfrac{1}{4}(\vec{r}_i + \vec{r}_j - \vec{r}_k - \vec{r}_l)^2, \quad (15)$$

the energy-energy correlation length (second moment definition) is

$$\xi_E^2 \equiv \frac{\sum_{j,k,l}\alpha_{0,j}\alpha_{k,l} r_{0,j,k,l}^2(\langle\phi_0\phi_j\phi_k\phi_l\rangle - \langle\phi_0\phi_j\rangle\langle\phi_k\phi_l\rangle)}{\sum_{j,k,l}\alpha_{0,j}\alpha_{k,l}(\langle\phi_0\phi_j\phi_k\phi_l\rangle - \langle\phi_0\phi_j\rangle\langle\phi_k\phi_l\rangle)}, \quad (16)$$

and the second moment definition of the spin-spin correlation length is,

$$\xi^2 \equiv \frac{\sum_n r_{0,n}^2\langle\phi_0\phi_n\rangle}{\sum_n\langle\phi_0\phi_n\rangle}, \quad (17)$$

where $r_{0,n}^2 \equiv (\vec{r}_0 - \vec{r}_n)^2$. Note that this definition is a factor of $2d$ larger that the usual one in order to conform to the work of Cardy,[3] where d is the spatial dimension. Now we observe that if, in line with the index values allowed by the α's when we do the sums in (16), we write $\vec{r}_j = \vec{r}_0 + \vec{\delta}_j$ and $\vec{r}_l = \vec{r}_k + \vec{\delta}_l$ where the $\vec{\delta}$'s are unit vectors, then from (15) we may write,

$$r_{i,j,k,l}^2 = r_{0,k}^2 + (\vec{r}_0 - \vec{r}_k)\cdot(\vec{\delta}_j - \vec{\delta}_l) + \tfrac{1}{4}(\vec{\delta}_j - \vec{\delta}_l)^2. \quad (18)$$

When we remember that we will sum in a symmetrical manner over k we may drop the dot product in (18) as the the $+k$ and $-k$ corresponding terms will cancel in pairs. The magnitude of the remaining term in (18) involving the $\vec{\delta}$'s is always less than or equal to unity. Thus $r_{i,j,k,l}^2 \leq r_{0,k}^2 + 1$ and so by Giffiths' second inequality[11] we get an upper bound on ξ_E^2 by replacing $r_{i,j,k,l}^2$ by $r_{0,k}^2$ and adding unity.

The next ingredient that we need is the result of Aizenman[10] (Prop. 5.1),

$$\begin{aligned}\langle\phi_0\phi_j\phi_k\phi_l\rangle - \langle\phi_0\phi_j\rangle\langle\phi_k\phi_l\rangle) = \\ \langle\phi_0\phi_k\rangle\langle\phi_j\phi_l\rangle(1 - \text{Prob.}(0,k,j,l)) \\ + \langle\phi_0\phi_l\rangle\langle\phi_j\phi_k\rangle(1 - \text{Prob.}(0,l,j,k)), \quad (19)\end{aligned}$$

where Prob. is a probabilty depending on its arguments as explained by Aizenman. Now because of the symmetry of the sums, we get the same contribution to ξ_E from each of the two terms on the right-hand side of (19), so we need only take one of them. Thus we have,

$$\xi_E^2 \leq 1 + \quad (20)$$

$$\frac{\sum_{j,k,l}\alpha_{0,j}\alpha_{k,l} r_{0,k}^2[\langle\phi_0\phi_k\rangle\langle\phi_j\phi_l\rangle(1 - \text{Prob.}(0,k,j,l))]}{\sum_{j,k,l}\alpha_{0,j}\alpha_{k,l}[\langle\phi_0\phi_k\rangle\langle\phi_j\phi_l\rangle(1 - \text{Prob.}(0,k,j,l))]}.$$

If we subtract (20) from (17), we will get,

$$\begin{aligned}(\xi^2 + 1 - \xi_E^2)\left\{\sum_{j,k,l}\alpha_{0,j}\alpha_{k,l}[\langle\phi_0\phi_k\rangle\langle\phi_j\phi_l\rangle \right. \\ \left. \times(1 - \text{Prob.}(0,k,j,l))]\right\}\left\{\sum_n\langle\phi_0\phi_n\rangle\right\} \geq \\ \sum_{j,k,l,n}\alpha_{0,j}\alpha_{k,l}\langle\phi_0\phi_k\rangle\langle\phi_j\phi_l\rangle\langle\phi_0\phi_n\rangle \\ \times(1 - \text{Prob.}(0,k,j,l))(r_{0,n}^2 - r_{0,k}^2). \quad (21)\end{aligned}$$

Now for clarity of exposition, let us define,

$$X(0,k) \equiv \sum_{j,l}\alpha_{0,j}\alpha_{k,l}\langle\phi_j\phi_l\rangle(1 - \text{Prob.}(0,k,j,l)). \quad (22)$$

The right-hand side of (21), becomes

$$\begin{aligned}\sum_k[X(0,k)]\langle\phi_0\phi_k\rangle\sum_n[r_{0,n}^2]\langle\phi_0\phi_n\rangle \\ - \sum_k[X(0,k)r_{0,k}^2]\langle\phi_0\phi_k\rangle\sum_n[1]\langle\phi_0\phi_n\rangle, \quad (23)\end{aligned}$$

which we recognize as being proportional to the negative of the correlation coefficient of $r_{0,k}^2$ and $X(0,k)$, against $\langle\phi_0\phi_k\rangle$ as a measure by Griffiths' first inequality.[11] As $r_{0,k}^2 \to \infty$ when $k \to \infty$, as $X(0,k)$ is driven to zero by the $\langle\phi_j\phi_k\rangle$ terms for $K \leq K_c$, and $0 \leq$ Prob. ≤ 1 always, the correlation coefficient is negative, and so we conclude that

$$\xi^2 + 1 \geq \xi_E^2, \tag{24}$$

which concludes our demonstration that the replacement of ξ_E^2 by ξ^2+1 in (1) gives an upper bound for the central charge c.

If for the three-dimensional Ising model (simple cubic lattice) we substitute known values[12] in this bound for the hyper-universal amplitude ratio, which is the central charge estimator in two dimensions, then we compute that $c \leq 27.4 \pm 0.3$. In this example, the addition of the Baker-Bessis[13] bound on the decay of spin-spin correlation function allows us to lower the upper bound by a factor of 2^d which reduces the upper bound to 3.42, and leaves open the possibility that the true value may be in an interesting range.

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