Dimensional Expansions for the Ising Model

In a recent letter [1] Bender, Boettcher, and Lipatov make the interesting suggestion that quantum boson field theories can be usefully studied by an expansion in powers of the spatial dimension d. They point out that an important point in its favor is that the zero dimensional model is often exactly solvable. In order to get a better understanding of the nature of this expansion I will briefly discuss the Ising model on the hyper-simple-cubic set of lattices. In this context it is appropriate [2] to think of the Ising model as the lattice-cutoff, hyper-strong-coupling limit of the massive, ϕ^4_d, boson field theory. In a previous paper [3] I have reported the high temperature expansion for this model in $u = \tanh K$ through the ninth order, exact for all dimensions. The partition function is, $Z = \sum_{\sigma_{\mathbf{j}}=\pm 1} \exp[K \sum_{\mathbf{j}} \sum_{\boldsymbol{\delta}} \sigma_{\mathbf{j}} \sigma_{\mathbf{j}+\boldsymbol{\delta}}]$, where the sum over $\mathbf{j}$ is over the lattice and the sum over $\boldsymbol{\delta}$ is over half the set of nearest neighbors. One quantity of interest to study is the (reduced) magnetic susceptibility. I repeat here for the reader's convenience a slightly reorganized version of the series expansion for χ^{-1}. It is

$$\begin{aligned}1/\chi = 1 &- \tbinom{d}{1}(2u - 2u^2 + 2u^3 - 2u^4 + 2u^5 - 2u^6 + 2u^7 - 2u^8 + 2u^9 + \ldots) \\ &+ \tbinom{d}{2}(8u^4 - 16u^5 + 40u^6 - 80u^7 + 184u^8 - 368u^9 + \ldots) \\ &+ \tbinom{d}{3}(192u^6 - 480u^7 + 3168u^8 - 8608u^9 + \ldots) + \tbinom{d}{4}(10368u^8 - 27136u^9 + \ldots) + \cdots . \end{aligned} \tag{1}$$

We can see at once that the terms in (1) through the $\binom{d}{1}$ terms can be directly summed to give, $1/\chi = 1 - 2\binom{d}{1}u/(1+u) + \ldots$, which gives the exact result for $d = 1$, and gives the leading order in the $1/d$ expansion of u_c. The remaining terms of course vanish exactly for $d = 1$ by the properties of the binomial coefficients. Let us now consider the terms through $\binom{d}{2}$. As the rest of the terms vanish for $d = 2$, these terms must sum to the exact result for the two-dimensional Ising model. $1/\chi$ will vanish at $u = u_c = \sqrt{2} - 1$ like $(u_c - u)^{1.75}$, an analytic singularity. In the case of the susceptibility, this function is well defined for $u > u_c$, but for the magnetization, the function would be multivalued for $u > u_c$. Next, the $\binom{d}{3}$ terms must sum to what is necessary to give the result for the $d = 3$ Ising model. Since the critical point for $d = 3$ is about $u_c = 0.218135$, which is smaller than that for $d = 2$, we find that there is an analytic singularity at that point such that $1/\chi$ vanishes like a noninteger power. For the region beyond, the $\binom{d}{3}$ coefficient must compensate for the singularity in the $d = 2$ case, as the susceptibility is not singular there for the $d = 3$ case. The same will be true for the case of $d = 4$, but for $d \geq 5$ Aizenman and Fernández [4,5] have proven that classical values obtain, so we expect that in this range of dimension $1/\chi$ will behave like $|u - u_c|$, and we note $u_c \sim 1/(2d)$ as $d \to \infty$.

It is also useful to write the expression for $1/\chi$ in terms of the susceptibility χ_d at integer dimension d. It is

$$1/\chi = 1 + \sum_{j=1}^{\infty} \binom{d}{j} (-1)^j \left[\sum_{k=0}^{j} \binom{j}{k} (-1)^k / \chi_k \right], \qquad (2)$$

where $\binom{d}{j} = 0$ when d is an integer and $j > d$. We take $\chi_0 = 1$. We observe from (1) that the combination in (2) cancels identically for the first 9 powers of u for $j > 4$, as the sum $j = 1, \ldots, 4$ is exact for all d to this order. We see that the higher j terms only contribute to progressively higher powers of u, so that only a finite number of terms of the sum contribute to fixed order in u. Using,

$$\binom{d}{j} = (-1)^{j-1}/j \left[d - d^2 \sum_{k=1}^{j-1} (1/k) + \cdots \right], \qquad (3)$$

we can expand (2) in powers of d. To the order u^9 we get, directly from (1), $\chi^{-1} = 1 + d(-2u + 2u^2 - 2u^3 - 2u^4 + 6u^5 + 46u^6 - 122u^7 - 1662u^8 + 4096\frac{2}{3}u^9 + \ldots) + d^2(4u^4 - 8u^5 - 76u^6 + 200u^7 + 3260u^8 - 8317\frac{1}{3}u^9 + \ldots) + d^3(32u^6 - 80u^7 + 2064u^8 - 5349\frac{1}{3}u^9 + \ldots) + d^4(432u^8 - 1130\frac{2}{3}u^9 + \ldots)$.

Since $1/\chi$ vanishes linearly for all $d \geq 5$, it is very likely, that the series expansion in u here simply carries on in an analytic fashion right through u_c. That means, using (3) in (2) to represent $1/\chi$ in terms of χ_d, we see that every power of d has the singularities of the $d = 2,\ 3,\ 4$ cases in its coefficient. Since the case $d = 4$ has its u_c closest to $u = 0$, it is this singularity which sets the limit of useful, normal series analysis in u for the coefficients of the d expansion. As the structure of their approximate calculations is somewhat similar to those quoted above for the high-temperature expansion of the Ising model, I raise the question whether in field-theoretic applications, the method of [1] may be subject to similar limitations.

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