Renormalized coupling constant in the Ising model

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Abstract. The three dimensional Ising mode is studied by means of Swendsen-Wang cluster Monte Carlo method. The simulation is performed on finite size systems upto $L = 96$ and the renormalized coupling constant is estimated as $g^* = 25.0(5)$, in agreement with the conjecture $0 < \lim_{L\to\infty} \lim_{K\to K_c} g(K,L) < g^*$.

05.50.+q, 05.70.Jk,02.70.Lq,75.10.Hk

1. Introduction and Summary

For about a third of a century now there has been uncertainty about a fundamental question in the theory of critical phenomena. The question is whether the "hyperscaling hypothesis" is valid or not. We will report our results on this question for the three dimensional Ising model. (A preliminary report was given in reference [1].) The hyperscaling hypothesis relates to the relations between critical indices which depend on the spatial dimension. To understand what the question really is, we give a bit of background. The Ising model, of course, is defined by spin variables on a spatial lattice which can take on the values plus or minus unity. The Hamiltonian, H, for the model is the exchange energy J times the sum of the products of all the nearest-neighbor spins. We consider the ferromagnetic case where $J > 0$ and abreviate J/kT by K, where k is Boltzmann's constant and T is the temperature. An important set of objects to study in this model is the set of spin-spin correlations. We define them in terms of $S_A = \prod_{i\in A} s_i$, where the s_i are the individual spins and A is an index set. Then the spin-spin correlations $\langle S_A\rangle$ are the expectation values with respect to the Gibbs weight $\exp(-H/kT)/Z$, where Z is the partition function which is just the normalization for the Gibbs weight. These correlation functions have a number of important properties. Griffiths [2] showed that $\langle S_A\rangle \geq 0$, and (generalized by Ginbre [3]) $\langle S_A S_B\rangle - \langle S_A\rangle\langle S_B\rangle \geq 0$. In addition they possess [4] the cluster property, $\langle S_A S_B\rangle - \langle S_A\rangle\langle S_B\rangle \leq \mathrm{O}(\mathrm{e}^{-\mu^2\rho})$ where ρ is the distance between A and B and μ is defined by the two-point correlation function through the relation

$$\lim_{\boldsymbol{r}\to\infty} \langle s_{\boldsymbol{s}} s_{\boldsymbol{s}+\boldsymbol{r}} \rangle \propto \exp(-\mu^2 |\boldsymbol{r}|). \tag{1}$$

In (1) we consider the case where $K < K_c$, the critical value, as here $\langle s_i \rangle = 0$ and the two-point function decays exponentially [5]-[6]. In addition they have the property of "two-point dominance" as shown by the Lebowitz [7] four-point inequalities,

$$\langle s_i s_j s_k s_l \rangle - \langle s_i s_j \rangle \langle s_k s_l \rangle \le \langle s_i s_k \rangle \langle s_j s_l \rangle + \langle s_i s_l \rangle \langle s_j s_k \rangle, \tag{2}$$

and for higher-point correlations by Newman's [8] Gaussian inequalities. These are unusual inequalities in the sense that higher-point expectation values are bounded in terms of lower order ones. In (2), if spins i and j are close together but far from l and k, we have an example of the cluster property. To get ahead of the story, the key question turns out to be whether the inequalities like (2) saturate and become equalities or not.

To become more quantitative, let us define a few quantities. First the magnetic susceptibility is given by

$$\chi(K, L) = L^{-d} \sum_{i,j} \langle s_i s_j \rangle, \tag{3}$$

where for ease of exposition we consider the hyper-cubic lattice family with edge L in spatial dimension d. The critical point K_c is the smallest value of K for which χ diverges. It is at this point that the μ defined above goes to zero. It is convenient to define the correlation length, ξ which is related to μ^{-2}.

$$\xi^2(K, L) \equiv \frac{\sum_{\boldsymbol{r},\boldsymbol{s}} |\boldsymbol{r}|^2 \langle s_{\boldsymbol{s}} s_{\boldsymbol{s}+\boldsymbol{r}} \rangle}{2dL^d \chi}, \tag{4}$$

and also to define the second derivative of χ with respect to the magnetic field H as,

$$\frac{\partial^2 \chi}{\partial H^2} = L^{-d} \sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{t},\boldsymbol{u}} \left[\langle s_{\boldsymbol{r}} s_{\boldsymbol{s}} s_{\boldsymbol{t}} s_{\boldsymbol{u}} \rangle - \langle s_{\boldsymbol{r}} s_{\boldsymbol{s}} \rangle \langle s_{\boldsymbol{t}} s_{\boldsymbol{u}} \rangle - \langle s_{\boldsymbol{r}} s_{\boldsymbol{t}} \rangle \langle s_{\boldsymbol{s}} s_{\boldsymbol{u}} \rangle - \langle s_{\boldsymbol{r}} s_{\boldsymbol{u}} \rangle \langle s_{\boldsymbol{s}} s_{\boldsymbol{t}} \rangle \right]. \tag{5}$$

All of these quantities diverge at the critical point like some power of $(K_c - K)$. The conventional notation is

$$\chi \propto (K_c - K)^{-\gamma}, \quad \xi \propto (K_c - K)^{-\nu}, \quad \frac{\partial^2 \chi}{\partial H^2} \propto (K_c - K)^{-\gamma - 2\Delta}, \tag{6}$$

which defines the critical exponents or indices γ, ν, Δ. It is known [9]-[10] that these critical indices satisfy the inequality

$$\gamma + d\nu \ge 2\Delta. \tag{7}$$

If we take the idea [11, 12] that there is only one important length near the critical point and that it is ξ, and that everything is a function of the ratio of the distances to ξ, then we come to the conclusion that we should have an equality,

$$\gamma + d\nu = 2\Delta, \tag{8}$$

which is a *hyperscaling relation.* This equality is equivalent to the idea that the cluster property holds with a non-zero coefficient for the four-point, spin-spin correlation function, no matter how we pair up the variables. At this point it is worth while to remember a featue of the Gaussian model. (This model is the same as the Ising model, except that instead of the spin variables taking on only the values ± 1, all real values are taken with probablility $e^{-s^2/2}\, ds/\sqrt{2\pi}$.) In this model, the $\leq$ sign becomes and $\equiv$ sign in (2) and $\frac{\partial^2\chi}{\partial H^2} \equiv 0$. This result shows that there is, in principal, no restriction on reasonable models that prevents the leading order of the correlation functions from cancelling out. When this cancelation occurs, the the inequality sign is the correct one, and the hyperscaling hypothesis fails. Aisenman [13] has shown that it also fails for the Ising model for $d > 4$. On the other hand, for the two dimensional Ising model the hyperscaling hypothesis is valid [13, 14]. For the one-dimentional Ising model, equation (8), as appropriately modified to take account of the zero-temperature critical point, holds [15].

We mention that there is another hyperscaling relation involving the specific heat index α which converts the Josephson inequality [16], $d\nu \geq 2 - \alpha$ into an equality. The index α is defined by $C_H \propto (K_c - K)^{-\alpha}$ in the limit as $K \to K_c$. This relation fails [15] in one dimension, and can at best be said to hold weakly in two dimensions, because in two dimensions $\nu = 1$ and the specific heat at constant magnetic field C_H diverges like $\ln(K_c - K)$ which corresponds to an α of zero. All of these exponent relations correspond to the critical point limit of the logarithm of estimator functions. If instead we look at the estimator function, we expect it to approach a finite number at the critical point when hyperscaling holds. In the case of a logarithmic approach, this expectation is not fufilled, and the estimators function fail to satisfy the expected properties, but do not strictly speaking cause a failure in the exponent relations. We will not study this hyperscaling relation in this paper, except for some estimates of α and ν.

The interest in this question for $d = 3,\ 4$ was heightened with the introduction of the renormalization group theory of critical phenomena by Wilson [17], and for which he got the 1982 Nobel prize in physics. One of the most powerful computational tools for this theory is the field theory method with its expansion in variable dimension, i.e., the ϵ-expansion [18]. The hyperscaling hypothesis is implicitly assumed in the method and it would be a matter of extreme importance if it should fail, as it would have a deliterious effect on a very large number of computations which have been carried out using this method, not to mention the problem of a proper understanding of the physics which would be associated with such a failure.

A quantitative way to examine this question is by an examination of the "renormalized coupling constant," g^*. First we define the estimator function,

$$g(K, L) \equiv -\left(\frac{v}{a^d}\right) \frac{\partial^2\chi}{\partial H^2} \Big/ \chi^2 \xi^d, \tag{9}$$

from which the renormalized coupling constant is defined by

$$g^* = \lim_{K \to K_c - 0} \lim_{L \to \infty} g(K, L) \tag{10}$$

It follows from equation (7) that g^* does not diverge to infinity. If $g^* > 0$, then the hyperscaling relation (8) holds. If $g^* = 0$, then hyperscaling may fail. Hara and Tasaki [19] have proven in four dimensions that $g(K, \infty) \propto [n_0 + |\ln(K_c - K)|]^{-1/2}$ which means that $g^* = 0$. However, it vanishes logarithmically, so equation (8) holds, but only weakly. The remaining case is for $d = 3$, which is what we will investigate in this paper. The renormalized coupling constant g^* is estimated [39] as $g^* = 23.73(2)$ for the ϕ^4 model. **//Other references should be here?//**

In his 1967 review, Fisher [20] well summed up the then current status of the hyperscaling relations as "These relations involving the dimensionality directly seem most open to question, ..." In a subsequent series analysis Baker [21] estimated that $2\Delta - d\nu - \gamma \equiv -\omega^* \approx -0.028$. It is this type of counter hypothesis that has made this issue so difficult to resolve. It says that perhaps $g(K, \infty)$ vanishes as $K \to \infty$, but like a very small power of $(K_c - K)$. Practically speaking, in this case, the curve should show almost no deviation from one which tends to a constant limit until you are very close to the critical point, and then it drops precipitously. Thus direct computation of the $g(K, \infty)$ or $g(K, L)$ as $K \to K_c$, considering the practical limitation of such calculation, can not really demonstrate the validity of hyperscaling. In this paper, we take a two pronged approach to the resolution of this issue. First, we compute $g(K, L)$ directly by a Monte Carlo procedure for a sequence of system sizes so that we keep ξ/L fixed. We will argue that we have choosen a small enough value to keep systematic errors at, or below, the 1 % level. This method will provide a direct estimate of g^*, provided $\omega^* = 0$ without logarithmic corrections.

The point, $K = K_c$, $L = \infty$, is a very special point. We will show, when we are very specific about the definition of our estimator $g(K, L)$, that limit from the low-temperature side,

$$g^\ddagger = \lim_{K \to K_c + 0} \lim_{L \to \infty} g(K, L) = 0, \tag{11}$$

which, if hyperscaling is valid, is not equal to g^*. Thus this point is what is called a point of non-uniform approach. We will argue that the limit,

$$\hat{g} = \lim_{L \to \infty} \lim_{K \to K_c} g(K, L), \tag{12}$$

is a lower bound to g^* and our calculations show that $\hat{g}$ is distinctly greater than zero, and so $g^* > 0$, which in turn imply that the hyperscaling relation (8) is valid, which was our main point of inquiry. Note that the quantity $g^\ddagger$ is not the same as the renormalized coupling constant, g^*_- that is computed from a proper approach to the critical point from the low-temperature side, nor is it the ratio of the corresponding, critical-point amplitudes G_1^- taken on the low temperature side. These latter quantities are discussed by Zinn et al. [22].

The first clear illustration of the fact that $K = K_c$, $L = \infty$ is a point of non-uniform approach for $g(K, L)$ that we know of is that given by Baker [23] from exact calculations by the Markov property method for small two-dimensional squares with periodic boundary conditions. We reproduce his figure here as Figure 1. In this figure we see for $K < K_c$ that the finite size system results are approaching the infinite system size limit in a straightforward fashion. However for $K = K_c$, the values of $g(K_c, L)$ seem instead to be approaching a limit around 3 instead of the value of $g^* \approx 14.66 \pm 0.06$. It is this result which foreshadowed the present work.

From the practical point of view, another important insight was the recognition of Baker and Erpenbeck [24] of the data collapse which results from plotting the renormalized coupling $g(K, L)$ versus ξ_L/L. (See also, Kim and Landau [25].) The same data collapse does not occur if the plot is made of $g(K, L)$ versus ξ_∞/L for example. We reproduce Baker and Erpenbeck's figure here as Figure 2. In addition these author report a clear warning that care must be taken not to use too large a value of ξ_L/L. They demonstrate that $\xi_L/L = 0.26$ is too large for accurate work, which means that some of the prior efforts in the investigation of this area would likely suffer from significant systematic errors. Baker [23] had observed that for the two-dimensional systems he studied that $\xi_L/L \leq 1/(7 \pm 1)$ was required for 1 % accuracy. Baker and Erpenbeck further observed that the allowed value of ξ_L/L for say 1 % work seems to increase somewhat with L and concluded that for large systems the maximum allowed value of ξ_L/L must be somewhere in the range $0.11 - 0.26$. They recommended that $\xi_L/L \leq 0.10$ for work at the one-percent level.

2. Method of Calculation

For our computations, we used the Swendsen-Wang algorithm [26] for spin-updating. This type of algorithm has two advantages over the conventional algorithm, *i.e.*, reduction in the autocorrelation time, and reduction in the variances of equilibrium distributions of relevant quantities. As has been reported by several authors [27], we observed that the cluster algorithm with improved estimators dramatically reduces statistical errors. It was reported [28] that cluster algorithms are not necessarily much more efficient than conventional algorithms with multi-spin coding technique when only the first benefit, i.e., the reduction in the correlation time, is taken into account. We emphasize, however, that not only reduction of the correlation times but also use of improved estimators was crucial to the present work. In fact, our preliminary computation showed that, for the 64 cube, it was impossible to obtain results as accurate as the ones presented in this letter by means of a conventional algorithm within a reasonable computational time ($\sim$ a few months) and within the given resources, at the temperatures of the present interest. Our computations were performed on a cluster of eight IBM RS/6000 model 590's, SUN Sparc Stations, and ???. **//What machines did you use?//** As we will show, even for smaller lattices, it was obvious that the cluster algorithm performs better.

It is known[35] that a Monte Carlo simulation with a cluster algorithm, such as the one used in this paper, can be viewed as a Markov process in an extended configuration space that is a product of the original spin configuration space and a graph space. Various physical quantities defined in terms of spin variables have corresponding definitions in graphical terms as well. For instance, it is well-known that we can estimate the suseptibility, which is usually defined as the second moment of the total magnetization, as the average volume of clusters. It is also known that the equilibrium distributioin functions of two corresponding estimators, one in terms of spins and the other in terms of graphs, can have very different variances although the mean values are equal. In the above example, the graphical estimator is more advantageous because it has a much small variance than the estimator defined on spin configurations.

Since we will deal with the renormalized coupling constant, which is a product of several macroscopic quantities, we need to obtain the graphical representation of all quantities involved. Otherwise the relative error associated with the quantity estimated through the poor estimator would be much larger than the relative errors from other sources and it would dominate the relative error in the final estimate. First we rewrite the definition of the renormalized coupling constant (9) as

$$g(K,L) \equiv -\left(\frac{L}{\xi_L}\right)^d \frac{\langle M^4\rangle - 3\langle M^2\rangle^2}{\langle M^2\rangle^2} \tag{13}$$

Therefore, we need improved estimators for ξ_L, $\langle M^2\rangle$ and $\langle M^4\rangle$.

A new graphical estimator for the correlation length ξ_L is derived as follows. We define

$$f(\boldsymbol{k}) \equiv 4\left(\sin^2\frac{k_x}{2} + \sin^2\frac{k_y}{2} + \sin^2\frac{k_z}{2}\right)\left(1 - \frac{\chi(\boldsymbol{k})}{\chi}\right)^{-1}. \tag{14}$$

with

$$\chi(\boldsymbol{k}) \equiv \langle |M(\boldsymbol{k})|^2\rangle / N \tag{15}$$

where $M(\boldsymbol{k}) = \sum_{\boldsymbol{r}} \exp(-i\boldsymbol{k}\cdot\boldsymbol{r}) S_{\boldsymbol{r}}$. The quantity $f(\boldsymbol{k})$ converges to ξ_L^{-2} in the limit of $L \to \infty$ and $|k| \to 0$ with a correction term proportional to $|k|^2$. In order to eliminate this correction term, we formed a linear combination of $f(\boldsymbol{k})$ with six smallest possible values of $|\boldsymbol{k}|$ which correspond to the nearest and the second nearest neighbors to the origin in the reciprocal space:

$$\xi_L^{-2} = [2f(\Delta k,0,0) + 2f(0,\Delta k,0) + 2f(0,0,\Delta k)$$

$$- f(0,\Delta k,\Delta k) - f(\Delta k,0,\Delta k) - f(\Delta k,\Delta k,0)]/3, \tag{16}$$

where $\Delta k \equiv 2\pi/L$. This expression is correct upto the second order in $1/L$ and we estimated ξ_L through this. Thus estimation of ξ_L is reduced to computing $\langle M(\boldsymbol{k})\rangle$ for several smallest reciprocal vectors..

It is well known that an improved estimator for the second moment of magnetization is simply the average size of clusters [29], *i.e.*,

$$\langle M^2\rangle = \left\langle \sum_c V_c^2 \right\rangle. \tag{17}$$

Here, V_c is the number of sites in a cluster c. For the fourth moment of magnetization, we can derive by the same methods the corresponding estimator,

$$\langle M^4\rangle = 3\left\langle \left(\sum_c V_c^2\right)^2 \right\rangle - 2\left\langle \sum_c V_c^4 \right\rangle. \tag{18}$$

We can express $\chi(\vec{k})$ in a form analogous to (17).

$$\langle M^2(\boldsymbol{k})\rangle = \left\langle \sum_c V(\boldsymbol{k})_c^2 \right\rangle \tag{19}$$

with

$$V_c(\vec{k}) \equiv \left| \sum_{\vec{r}\in c} \exp(i\vec{k}\cdot\vec{r}) \right|. \tag{20}$$

Thus, we have expressed all the necessary quantities in terms of improved estimators.

Our simulation consists of $N_{\rm S}$ independent runs. Each run consists of $N_{\rm E}$ sweeps for equilibration followed by $N_{\rm M}(1+n_{rmR})$ Monte Carlo sweeps for measurement. Actual measurements are done every $(1+n_{\rm R})$ steps and therefore the number of total measurements performed in each run is $N_{\rm M}$. Accordingly, the total number of Monte Carlo sweeps performed in the whole simulation is $N_{\rm total} \equiv N_S[N_E+(n_R+1)N_M]$. (The item marked by $^{(a)}$ in Table I was performed in a different fashion from this. For this simulation, the equilibration sweeps were not performed for the second or later runs, but performed only at the beginning of the whole simulation. Therefore, N_E in Table I for the item refers only to the first run, so the total number of Monte Carlo sweeps here is just $N_E + N_S(n_R+1)N_M$.) In the conventional algorithm is it necessary to take $n_{\rm R} > 0$. One Monte Carlo sweep includes assignments of 'deletion' or 'freezing' to all bonds and attempts to flip all clusters. The numbers used in our computation are listed in Table I. Since the autocorrelation time is, regardless of the definition, less than 100 [30] up to the system size of 64^3, the numbers $N_{\rm E}$ and $N_{\rm M}$ listed in the table are large enough to exclude systematic error due to auto-correlation. As mentioned already, the temperature of the simulation is chosen so that the resulting correlation length becomes approximately $1/10$ of the system size. We used the value of K_c in multiplying the estimate $g(K)$ by the factor $(K/K_c)^{3/2}$. Adding this factor is appropreate because the quantity $g(K)$ has the asymptotic dependence on K such that $g(K) \sim K^(2/3)$. As the actual value of K_c, we can use almost any one of very accurate estimates available today, and the choce does not affect the present result in any significant way. For example, Tamayo and Gupta [34] quote 0.221655, Ferrenberg and Landau [31] quote 0.2216595 ± 26 and Guttmann [32] using series analysis quotes 0.221657 ± 12. We used Blöte et al's value $K_c = 0.2216546(10)$ [37].

3. Results for direct estiate of g^*

In order to estimate g^*, we choose for each system size L the temperature T so that ξ/L is fixed to be a constant R. In the present paper we take $R = 10$ for various reasons mensioned in the first section. In other words, we estimated $g(K, R\xi(K))$ in practice. In order to assure that we are controlling possible systematic errors, we have performed exact computations on the two-cube and the three-cube at temperatures which correspond to $\xi_L = 0.2$ and 0.3 respectively. These results were compared with series expansion results, which are very precise estimate for $g(K, \infty)$ in such a temperature range, and we found that the two-cube result is about 2.2% below and the three-cube one about 0.8% below the infinite systems series results. In addition, we have compared our very long run, highly accurate Monte Carlo results for the eight-cube (Table 1) with the series results. We find that it is about 0.2% below the unbiased Padé approximant (See, for example, [42].) estimate. We conclude from these comparisons that it is very likely that the systematic errors $|g(K, R\xi(K)) - g(K, \infty)|$ are less than one percent, and perhaps much better.

We have obtained results for $g(K, R\xi(K))$ for a sequence of temperatures with $R = L/\xi \approx 10$ for various temperatures. The parameters used in simulations are listed in Table 1. In Table 1, parameters used in simulations are shown.

We illustrate our results in Fig. 1. We conclude that $g^*(K_c - 0) = 25.0(5)$. It can be seen that the central extrapolation of ref. [40], **//George, please send me the latest data for the series analysis results so I can use them in the figure.//** which tends to zero, falls well below our present results. We believe that this method does not properly account for the leading subdominate behavior. In [41] it was seen that g^* lies above the limit of $g(K_c, L)$ as $L \to \infty$ in the two-dimensional Ising model. In [34] using the histogram method, it was found that the value of $g(K, L)$ falls very rapidly to 6-7 as $K \to K_c$. Combining these results with ours, we conclude that the value of g^* is greater than zero and so hyperscaling holds for the three-dimensional Ising model. Finally, in [36], $\hat{g}$ is estimated as $\hat{g} = 4.9(1)$.

//What do you think about the following comment?// We here conjecture that the $g(K, L)$ is a decreasing function of K when L is fixed. This conjecture is

L	K	N_S	N_E	N_M	n_R	N_{total}	ξ	Δ
8	0.14905	7	1,000	1×10^7	0	7.0×10^7	0.80005(2)	0.011
16	0.1916	35	2,000	2×10^5	0	7.1×10^6	1.6024(2)	0.11
32	0.2108	35	2,000	2×10^5	0	7.1×10^6	3.2300(5)	0.15
64	0.2180	28	1,000	1×10^5	0	2.8×10^6	6.5843(20)	0.33
96	0.2197	41	1,000	7×10^4	0	4.5×10^5	9.8309(31)	0.32
(a) 16	0.1916	40	70	2×10^4	6	5.6×10^6	1.6070(39)	1.8
(b) 16	0.1916	40	2,000	4×10^3	0	2.4×10^5	1.6033(14)	0.7

Table 1. The parameters used in the computation and the results. The value $K_c = 0.2216546$ given in [Bloete] was used. All the results presented are obtained through the cluster algorithm except for (a). The last column Δ is the estimated statistical error in $(K/K_c)^{3/2}g(K,L)$. The rows (a) and (b) are included only for comparison of the conventional algorithm (a) and the cluster algorithm (b).

L	$\langle M^2 \rangle$	$\langle M^4 \rangle - 3\langle M^2 \rangle^2$	$\langle E^2 \rangle - \langle E \rangle^2$	g	$(K/K_c)^{3/2}g$
8	$4.20700(7) \times 10^0$	$-4.113(2) \times 10^2$	4.1554(6)	45.38(2)	25.02(1)
16	$1.2776(2) \times 10^1$	$-2.143(9) \times 10^4$	6.092(4)	31.9(1)	25.6(1)
32	$4.5681(8) \times 10^1$	$-1.92(1) \times 10^6$	9.224(7)	27.3(2)	25.4(2)
64	$1.7752(6) \times 10^2$	$-2.29(3) \times 10^8$	13.55(2)	25.5(3)	24.9(3)
96	$3.860(1) \times 10^2$	$-3.61(5) \times 10^9$	16.45(2)	25.5(3)	25.2(3)

Table 2. The estimates for various quantities.

suggested by Fig. ??. (Note that ξ_L/L increases as K increases and that if we exclude the factor $(K/K_c)^{3/2}$ the decreasing behavior would be even more enhanced.) Especially, for K smaller than K_c

$$g(K,L) \geq \lim_{K \to K_c - 0} g(K,L). \tag{21}$$

By taking the limit of $L \to \infty$ of both sides,

$$\lim_{L \to \infty} g(K,L) \geq g^{\ddagger}. \tag{22}$$

Then, by taking the limit of $K \to K_c - 0$ of both sides, we have

$$g^* \geq g^{\ddagger}. \tag{23}$$

Thus, $g^{\ddagger}$ gives a lower bound for g^*. Therefore, with the above conjecture and the finite estimates of $\hat{g}$, we expect that g^* is finite and that the hyper-scaling holds for the three-dimensional Ising model. Our finite estimate for g^* is further confirmation of this result although it is larger than the field theoretic esitimated quoted above. We

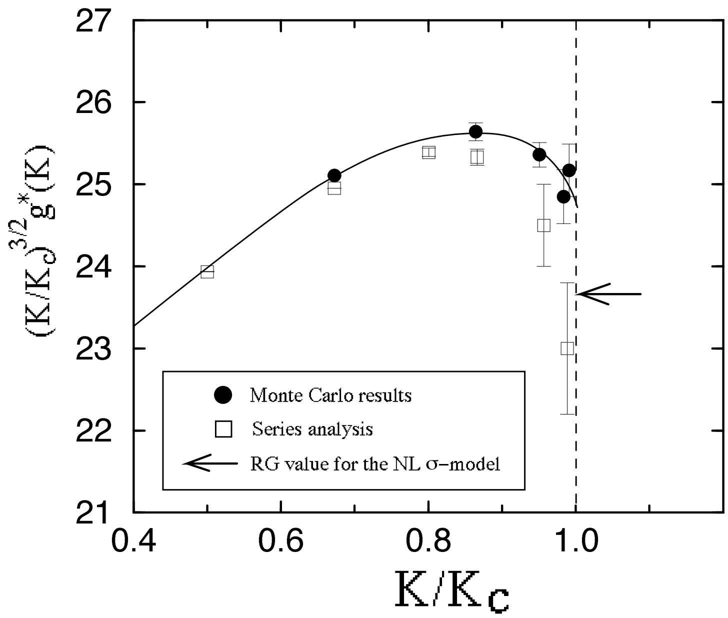

Figure 1. The renormalized coupling constant $g(K, L)$ times $(K/K_c)^{3/2}$ vs. K/K_c. The solid circles correspond to the simulations for $L = 8, 16, 32, 64$ and 96, from the left to the right, respectively. The curve is a mere guide line to eyes. The error bars represents one standard deviation. The series analysis results are from the same type of analysis as in the work of [BK81]. Cross comparison with other analyses shows that realistic errors on this series extrapolation are sufficiently large so as not to exclude our current Monte Carlo results. The field-theoretic renormalization group value is indicated by an arrow in the figure.

remind the reader of the caution of Nickel [38] who found non-analytic corrections to the Callan-Symanzik beta function $\beta(g)$ in one dimension, and suggested that there may also be such in other dimensions which would adversely effect the quoted error estimates for the field theory results.

4. Conclusion

We have estimated $g^*(K)$ which converges to the renormalized coupling constant as $T \to T_c + 0$ for various systems sizes upto $L = 96$. We found that $g^*(K)$ does not decreases down to zero as the critical temperature is approached as was conjectured from the previous result on $g^\ddagger$. The extrapolated value is $g^*(K_c - 0) = 25.0(5)$. We take this as a firm evidence for the validity of the hyperscaling relation in the three-dimensional Ising model.

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Figure captions

A plot of $Kg(K, L)/K_c$ versus K/K_c for the two-dimensional Ising model. The unlabeled curve is the series result for an infinite system, and the labels L indicate the curves for $L \times L$ square lattices with periodic boundary conditions. A plot of $g(K, L)(K/K_c)^{3/2}$ for the three dimensional Ising model for the simple cubic lattice with periodic boundary conditions. The cases shown are for systems of $L \times L \times L$ spins, and the plot is versus ξ_L/L. The point $\xi_L = 0$ is common for all values of $L = N$ and is exact.